PERGAMON Computers and Mathematics with Applications 42 (2001) 655–669

An International Journal
computers & mathematics
with applications

www.elsevier.nl/locate/camwa

A Problem of Hartman and Wintner: Approximation for Discrete Perturbations

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Abstract—Consider the second-order self-adjoint difference equation $\Delta(c_n\Delta x_n) + (a_n + f_n) x_{n+1} = 0$ as a perturbation of the eventually disconjugate difference equation $\Delta(c_n\Delta z_n) + a_n z_{n+1} = 0$, where $c_n \neq 0$. Asymptotic approximation for the fundamental system of solutions of the perturbed equation are expressed explicitly in terms of the coefficients and the principal (or recessive) solution of the unperturbed equation. In particular, the coefficient c_n is allowed to be oscillatory, and we do not assume absolute summability conditions on f_n. Results obtained are applied to a second-order Poincaré difference equation whose unperturbed equation has a double characteristic root. Nonlinear and nonhomogeneous perturbations are also considered. An example is given for illustration. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords—Asymptotic expression, Self-adjoint difference equation.

1. INTRODUCTION

We consider the second-order self-adjoint difference equation

$$\Delta(c_n\Delta x_n) + (a_n + f_n)x_{n+1} = 0, \qquad n \geq 0, \tag{1.1}$$

as a perturbation of the equation

$$\Delta(c_n\Delta z_n) + a_n z_{n+1} = 0, \qquad n \geq 0, \tag{1.2}$$

where Δ is the forward difference operator, $\Delta x_n = x_{n+1} - x_n$, c_n, a_n, f_n, $n = 0, 1, 2, \dots$, are sequences of real numbers with $c_n \neq 0$. We will find smallness conditions on f_n such that solutions of (1.1) and (1.2) behave asymptotically the same as $n \to \infty$.

This is a discrete version of the Hartman-Wintner problem of asymptotic equivalence between two differential equations

$$(c(t)x')' + (a(t) + f(t))x = 0, \qquad t \geq 0, \tag{1.3}$$

$$(c(t)z')' + a(t)z = 0, \qquad t \geq 0, \tag{1.4}$$

This research was supported by the NNSF of China under Grant 10071043 and partially by the National Key Basic Research Special Fund (973) (No. G1998020300).

PII: S0898-1221(01)00185-7

under the assumption that $c(t) > 0$ and (1.4) is nonoscillatory. The idea of this problem is to establish connections between fundamental systems of solutions of (1.3) and (1.4) of which the latter is supposed to be standard or simpler. Classical results on asymptotic integration of (1.3) in terms of its coefficients and the fundamental solutions of (1.4) were obtained by Hartman and Wintner in the 1950s [1, p. 379], and later significantly improved by Trench [2], Šimša [3], and Chen [4–6]. For more information in this direction, see [7].

However, for difference equations (1.1) and (1.2) to be nonsingular, it is natural to assume that $c_n \neq 0$ and discuss disconjugacy instead of nonoscillation.

If there exists a solution $z = \{z_n\}_{n=0}^{\infty}$ of (1.2) such that $c_h z_h z_{h+1} \leq 0$ and $c_k z_k z_{k+1} \leq 0$ with $z_{h+1} \neq 0$, $z_{k+1} \neq 0$, $h \neq k$, and $z_n \neq 0$ for n in the interior of the smallest interval containing $[h, h+1)$, $[k, k+1)$, then the two intervals $[h, h+1)$ and $[k, k+1)$ are called *conjugate intervals* of (1.2). For positive integers M and N with $M < N$, equation (1.2) is called *disconjugate* on the real interval $[M-1, N]$, if the interval $[M-1, N]$ contains no pair of conjugate intervals. Equation (1.2) is called *eventually disconjugate* or *disconjugate at* ∞ if (1.2) is disconjugate in a neighborhood of ∞, say, $[M, \infty)$. A discrete version of the Sturm separation theorem for (1.1) or (1.2) holds (cf. [8, Theorem 3.1], where more general matrix difference equations are considered). So, if (1.2) is eventually disconjugate and z_n is any nontrivial solution, then there is an integer $N \geq 0$ such that $c_n z_n z_{n+1} > 0$ for $n \geq N$. Clearly, in the case $c_n > 0$, eventual disconjugacy is equivalent to the usual concept of nonoscillation. For oscillation theory of linear or nonlinear difference equations, the reader is referred to [10,11].

A solution z_n of (1.2) is called *principal* or *recessive* at ∞, if there is an $N \geq 0$ such that

$$c_n z_n z_{n+1} > 0, \qquad n \geq N, \tag{1.5}$$

and

$$\sum_{n=N}^{\infty} \frac{1}{c_n z_n z_{n+1}} = \infty. \tag{1.6}$$

Otherwise, z_n is called *nonprincipal* or *dominant* if (1.5) holds and the series in (1.6) converges. For terminology, see [8,9]. It is shown that if (1.2) is eventually disconjugate, then there exists a principal solution z_n which is essentially unique (up to constant multiples) and any solution independent of z_n is nonprincipal.

Throughout this paper, we always assume that (1.2) is eventually disconjugate and z_n is its principal solution satisfying (1.5) and (1.6). The purpose of this paper is to find conditions on f_n so that (1.1) is eventually disconjugate and has a principal solution x_n satisfying

$$x_n = z_n(1 + o(1)), \qquad \text{as } n \to \infty, \tag{1.7}$$

where $o(1)$ is expressed explicitly in terms of the coefficients and z_n. The approximation for nonprincipal solutions can also be obtained from (1.7).

Clearly, a self-adjoint difference equation of form (1.2) can be written as a three term recurrence relation

$$c_{n+1} z_{n+2} + (a_n - c_n - c_{n+1}) z_{n+1} + c_n z_n = 0. \tag{1.8}$$

Conversely, given a difference equation of the form

$$\alpha_n z_{n+2} + \beta_n z_{n+1} + \gamma_n z_n = 0 \tag{1.9}$$

with $\alpha_n \gamma_n \neq 0$, multiplying through by $\tau_n = (1/\gamma_0) \prod_{k=0}^{n-1} (\alpha_k / \gamma_{k+1})$, one reduces (1.9) to (1.2) with $c_n = \prod_{k=0}^{n-1} (\alpha_k / \gamma_k)$ and $a_n = \beta_n \tau_n + c_n + c_{n+1}$.

Recently, classical WKB approaches have been applied to approximation for perturbations of linear difference equations (see, e.g., [12,13]). In [12], Geronimo and Smith obtain WKB approximation results for (1.9) where $\alpha_n \to 1$, $\gamma_n = 1$, and β_n keeps away from the subset $\{-2, 2\}$

of the complex plane, under some absolute summability conditions on the coefficients and their differences. Roughly speaking, we may regard (1.9) in [12] as a perturbation of a difference equation with constant coefficients whose characteristic polynomial has two distinct roots. Spigler and Viannello [13] study the WKB-type approximation for (1.1) in the case where $c_n = 1$, $a_n = 0$, and the second moment of $|f_n|$ is finite, i.e., $\sum^\infty n^2|f_n| < \infty$.

In the case of (1.1), where $a_n = 0$ and c_n is positive and bounded, i.e., $0 < m \le c_n \le M$ for two positive constants $m < M$, Chen and Wu [14] apply Riccati techniques to approximation of solutions of (1.1) under some summability (perhaps conditional) assumptions on f_n. However, in many important applications, c_n is not bounded above nor away from zero. Wei [15] studies the asymptotic approximation of solutions of (1.1) where c_n is positive and sometimes bounded. In this paper, we will assume $c_n \neq 0$ only. Our main tool is the second-level Riccati equation of (1.1) with respect to (1.2) (see definition in Section 2), established by Chen in [16]. The main results will be given in Section 2. An illustrative example will be shown in Section 3. Section 4 will apply the main results to a second-order Poincaré difference equation whose unperturbed equation has a double characteristic root, and Section 5 will be devoted to nonlinear or nonhomogeneous perturbations.

2. MAIN RESULTS

Throughout this paper, a sequence, say, $\{f_n\}_{n=0}^\infty$ will sometimes be denoted by a single letter f. Similarly, the sequence $\{f_n/g_n\}_{n=0}^\infty$ will be denoted by f/g, $\{f_n^2\}_{n=0}^\infty$ by f^2, etc. An empty sum will be treated as 0 and an empty product as 1. Landau's notations $O(\cdot)$ and $o(\cdot)$ will be referred to orders as $n \to \infty$. For simplicity, we will write $f \in \mathcal{C}$ if the limit of $\sum_{k=0}^{n-1} f_k$ as $n \to \infty$ exists and is finite. For convenience to the reader, we list two formulas involving the difference operator Δ, the latter of which is due to Abel

$$\Delta(a_n b_n) = b_n \Delta a_n + a_{n+1} \Delta b_n, \tag{2.1}$$

$$\sum_{k=n}^{m-1} a_k \Delta b_k = a_m b_m - a_n b_n - \sum_{k=n}^{m-1} b_{k-1} \Delta a_k, \qquad m > n. \tag{2.2}$$

Let z_n be a principal solution of (1.2) such that (1.5) and (1.6) hold. Let $x_n = z_n \xi_n$ in (1.1). Making use of (2.1) twice, with some manipulation, we have

$$z_{n+2}\Delta(c_n \Delta \xi_n) + c_n \Delta\xi_n (z_{n+2} - z_n) + f_n z_{n+1} \xi_{n+1} = 0,$$

and furthermore, multiplying through by z_{n+1} and using (2.1) again yields

$$\Delta(r_n \Delta \xi_n) + p_n \xi_{n+1} = 0, \tag{2.3}$$

where $p_n = z_{n+1}^2 f_n$, $r_n = c_n z_n z_{n+1} > 0$ for $n \ge N$ by (1.5), and $\sum^\infty 1/r_n = \infty$ by (1.6). Since

$$c_n x_n x_{n+1} = c_n z_n z_{n+1} \xi_n \xi_{n+1} = r_n \xi_n \xi_{n+1},$$

one easily sees that eventual disconjugacy of (1.1) is equivalent to nonoscillation of (2.3) and x is a principal solution of (1.1) if and only if $\xi = x/z$ is a principal solution of (2.3). Thus, for existence of a principal solution x of (1.1) satisfying (1.7), all that we need is to find conditions on p and r so that (2.3) is nonoscillatory and has a principal solution

$$\xi_n = 1 + o(1), \qquad \text{as } n \to \infty. \tag{2.4}$$

Without loss of generality, in the rest of the paper, we will assume that $r_n > 0$ for $n \ge 0$ and

$$R_n := \sum_{k=0}^{n-1} \frac{1}{r_k} \to \infty, \qquad \text{as } n \to \infty. \tag{2.5}$$

Suppose that $p \in \mathcal{C}$ and denote

$$P_n = \sum_{k=n}^{\infty} p_k. \tag{2.6}$$

Suppose that (2.3) is nonoscillatory and ξ is a nonoscillatory solution such that $\xi_n \xi_{n+1} > 0$ for $n \geq N \geq 0$. Let

$$v_n = \frac{r_n \Delta \xi_n}{\xi_n} - P_n. \tag{2.7}$$

Then, by (2.1), $\Delta v_n = \quad r_n \Delta \xi_n \Delta(1/\xi_n) - -(r_n \Delta \xi_n)^2/(r_n \xi_n \xi_{n+1})$. Since, from (2.7),

$$\frac{r_n \xi_{n+1}}{\xi_n} = \frac{r_n \Delta \xi_n}{\xi_n + r_n} = v_n + P_n + r_n,$$

we have

$$\Delta v_n + \frac{P_n}{r_n}(v_n + v_{n+1}) + \frac{v_n v_{n+1}}{r_n} + \frac{P_n^2}{r_n} = 0. \tag{2.8}$$

Motivated by the variation of constant method, we assume a solvability condition that there exists an $N \geq 0$ such that

$$|P_n| < r_n, \qquad \text{for } n \geq N, \tag{2.9}$$

which will be automatically fulfilled in our main results. Then the linear homogeneous part of (2.8) has a fundamental solution

$$q_N = 1, \quad q_n = \prod_{k=N}^{n-1} \frac{r_k - P_k}{r_k - P_k}, \qquad n \geq N+1, \tag{2.10}$$

and the Green function is

$$G(k,n) = g_k q_n = \prod_{j=n}^{k-1} (r_j - P_j) \prod_{j=n}^{k} \frac{1}{r_j - P_j}, \qquad k \geq n \geq N, \tag{2.11}$$

where

$$g_k = \frac{1}{q_k (r_k - P_k)}. \tag{2.12}$$

If $gP^2 \in \mathcal{C}$, we set

$$\widetilde{P}_n = \sum_{k=n}^{\infty} G(k,n) P_k^2. \tag{2.13}$$

Suppose that $p \in \mathcal{C}$ and (2.9) holds for some $N \geq 0$. It is shown [16, Theorem 3.2] that (2.3) is nonoscillatory or (1.1) is eventually disconjugate, if and only if $gP^2 \in \mathcal{C}$ and there exists a sequence $v = \{v_n\}_{n=N}^{\infty}$, $v_n \geq 0$, satisfying

$$v_n = \sum_{k=n}^{\infty} G(k,n) v_k v_{k+1} + \widetilde{P}_n. \tag{2.14}$$

REMARK 1. In Theorem 3.2 of [16], the coefficient p satisfies a weaker condition than $p \in \mathcal{C}$, namely, $\sum_{k=0}^{n-1} p_k$ may diverge as $n \to \infty$ but its weighted average, say, $R_n^{-1} \sum_{k=0}^{n-1} (1/r_k) \sum_{j=0}^{k-1} p_j$ has a finite limit C as $n \to \infty$, and P_n is defined by $P_n = C - \sum_{k=0}^{n-1} p_k$. Indeed, all the results below remain true whenever the convergence of $\sum^{\infty} p_n$ is replaced with the above-mentioned weaker condition.

Equation (2.14) is called the *second-level Riccati equation* of (2.3) or the second-level Riccati equation of (1.1) with respect to (1.2). See [16] for terminology. We will find conditions such that (2.14) has a solution v that is nonnegative and in some sense small, and then obtain the asymptotic approximation of the principal solution of (2.3) by solving (2.7) for ξ_n. To this end, we introduce some notations.

Two solutions x and y of (1.1) are called a fundamental system of solutions of (1.1) if they are linearly independent and

$$c_n(x_n y_{n+1} - x_{n+1} y_n) = 1. \tag{2.15}$$

Let f be any sequence. Denote the weighted average of the first n terms of f by

$$\overline{f}_n = \frac{1}{R_n} \sum_{k=0}^{n-1} \frac{f_k}{r_k}, \qquad n \geq 1.$$

Obviously, if $f_n \to 0$, then $\overline{f}_n \to 0$ as $n \to \infty$. If f is nonnegative and nonincreasing, then $f_n \leq \overline{f}_n$. Suppose that $p \in \mathcal{C}$. If $P/r \in \mathcal{C}$, denote

$$Q_n = \sum_{k=n}^{\infty} \frac{P_k}{r_k}.$$

THEOREM 2.1. *Suppose that $p \in \mathcal{C}$, $P/r \in \mathcal{C}$.*

(i) *If*

$$\varphi_n := \sum_{k=n}^{\infty} \frac{R_{k+1} P_k^2}{r_k} < \infty, \tag{A1}$$

then (1.1) is eventually disconjugate and has a fundamental system of a principal solution x satisfying

$$x_n = z_n \left(1 - Q_n + \frac{1}{2} Q_n^2 (1 + o(1)) + O(\varphi_n)\right), \tag{2.16}$$

$$\Delta\left(\frac{x_n}{z_n}\right) = \left(\frac{P_n}{r_n}\right) \left(1 - Q_n + O\left(Q_n^2\right)\right) + O\left(\frac{\varphi_n}{r_n R_n}\right), \tag{2.17}$$

and a nonprincipal solution y satisfying

$$y_n = z_n R_n \left(1 - Q_n + \overline{Q}_n + \overline{Q}_{n+1} + O\left(\overline{Q_n^2} + \overline{\varphi}_n\right)\right). \tag{2.18}$$

(ii) *Conversely, if $r_n \geq c > 0$ for some constant c and for all $n \geq 0$, and if (1.1) has a solution x such that $x_n / z_n \to 1$ as $n \to \infty$, then (A1) holds.*

REMARK 2. Theorem 2.1 generalizes and improves [14, Theorem 2.1] ($a_n = 0$, c_n is positive and bounded) and [15, Theorem 3.1] (c_n is positive and bounded above).

We note that for the principal solution x of (1.1), since x/z is bounded, estimates (2.16) and (2.17) are reasonably satisfactory. But, since for the nonprincipal solution y, $y/z \sim R$ is unbounded, more detailed asymptotic properties of y/z should be specified as follows:

$$\frac{y_n}{z_n} = R_n(1 + o(1)), \tag{2.19}$$

$$\Delta\left(\frac{y_n}{z_n}\right) = \frac{1 + o(1)}{r_n}, \tag{2.20}$$

$$\frac{y_n}{z_n} - R_n \to \text{const.}, \qquad \text{as } n \to \infty. \tag{2.21}$$

Each of the above properties indicates $y/z \sim R$, but they differ in degrees. Obviously, (2.20) implies (2.19), and usually (2.21) is stronger than (2.20). The nonprincipal solution y in Theorem 2.1 satisfies (2.19). But, for y to have property (2.20), we need one more assumption which turns out to be necessary as well

$$R_n P_n \to 0, \qquad \text{as } n \to \infty. \tag{A2}$$

THEOREM 2.2. *In Theorem 2.1, if in addition, (A2) holds, then the nonprincipal solution y of (1.1) satisfies*

$$\Delta\left(\frac{y_n}{z_n}\right) = \frac{1}{r_n}\left(1 + Q_n + \frac{1}{2}\,Q_n^2(1+o(1)) + O(\varphi_n + R_n|P_n|)\right). \tag{2.22}$$

Conversely, if the nonprincipal solution y satisfies $r_n\Delta(y_n/z_n) \to 1$ as $n \to \infty$, then (A2) holds.

REMARK 3. Theorem 2.2 reduces to [15, Theorem 3.2] if c_n is positive and bounded, and to [14, Theorem 2.2] if, in addition, $a_n = 0$.

For property (2.21) of the nonprincipal solution, we need more assumptions

$$S_n := \sum_{k=n}^{\infty} \frac{Q_k}{r_k} \text{ converges,} \tag{A3}$$

$$\omega_n := \sum_{k=n}^{\infty} \frac{R_k P_k}{r_k} \text{ converges,} \tag{A4}$$

$$\psi_n := \sum_{k=n}^{\infty} \frac{R_{k+1}^2 P_k^2}{r_k} < \infty. \tag{A5}$$

Evidently, (A5) $\Rightarrow$ (A1). It is easy to prove that if (A4) holds, then $R_nQ_n \to 0$ as $n \to \infty$ and

$$T_n := \sum_{k=n}^{\infty} \frac{Q_{k+1}}{r_k} \text{ converges,} \tag{2.23}$$

and $\omega_n = R_nQ_n + T_n$. If (A3) and (A4) hold, then $S_n = T_n + \sum_{k=n}^{\infty} P_k/r_k^2$, that is, (A3) and (A4) imply that $P/r^2 \in \mathcal{C}$ by (2.23). It should be mentioned here that in many cases, e.g., in the case $P_n \geq 0$ or $P_n \leq 0$, (A4) alone implies $P/r^2 \in \mathcal{C}$ and (A3). But, in general, since r is unbounded, (A3) and (A4) are different.

THEOREM 2.3. *Suppose that $p \in \mathcal{C}$, $P/r \in \mathcal{C}$. If (A3)–(A5) hold, then (1.1) is eventually disconjugate and has a fundamental system of a principal solution x satisfying (2.16),(2.17) and a nonprincipal solution y satisfying*

$$y_n = z_n\left(R_n + L - R_nQ_n - S_n - T_n + o(\|RQ\|_n) + O(\psi_n)\right), \tag{2.24}$$

where L is a constant and $\|RQ\|_n = \sup_{k\geq n} |R_kQ_k|$.

REMARK 4. Theorem 2.3 generalizes [14, Theorem 2.3] and [15, Theorem 3.3]. If $c_n = 1$, $a_n = 0$, Spigler and Vianello [13, Theorem 2.2] assume

$$\sum_{k=n}^{\infty} k^2|f_k| < \infty \tag{2.25}$$

and obtain two solutions of (1.1), $x_n = 1 + \varepsilon_n^{(1)}$ and $y_n = n + \varepsilon_n^{(2)}$ with $|\varepsilon_n^{(i)}| \leq V_n^{(i)}/(1 - V_n^{(i)})$, $i = 1, 2$, where

$$V_n^{(1)} = \sum_{k=n}^{\infty} (k - n + 1)|f_k|, \qquad V_n^{(2)} = \sum_{k=n}^{\infty} k(k - n + 1)|f_k|.$$

To compare condition (2.25) with ours, we note that, in this case, $R_n = n$, $P_n = \sum_{k=n}^{\infty} f_k$, and

(2.25) implies $n^2 P_n \to 0$, and so,

$$\begin{aligned}
Q_n &= \sum_{k=n}^{\infty} (k-n+1) f_k \le V_n^{(1)}, \\
\varphi_n &= \sum_{k=n}^{\infty} (k+1) P_k^2 = -n(n+1)\frac{P_n^2}{2} + \sum_{k=n}^{\infty} (k+1)(k+2)(P_k + P_{k+1})\frac{f_k}{2} \\
&\le \sum_{k=n}^{\infty} k^2 (|P_k| + |P_{k+1}|)|f_k| = o\left(V_n^{(1)}\right), \\
\psi_n &= \sum_{k=n}^{\infty} (k+1)^2 P_k^2 \le \sum_{k=n}^{\infty} k^3 (|P_k| + |P_{k+1}|)|f_k| = o\left(V_n^{(2)}\right).
\end{aligned}$$

Thus, (2.25) $\Rightarrow$ (A5) and (A2). It is also seen that (A3) and (A4) hold and all the theorems in this section are valid. In other words, we obtain stronger conclusions under weaker conditions. In this connection, we can also see the improvements in our results from the example given in the next section.

Since the proofs of Theorems 2.1–2.3 are technical and similar to those of [14, Theorems 2.1–2.3] and [15, Theorems 3.1–3.3], respectively, here we only sketch the ideas of proving and give a complete proof of Theorem 2.1(ii).

First of all, conditions given in each of the above theorems ensure nonoscillation of (2.3). By [16, Theorem 3.2], the second-level Riccati equation (2.14) has a solution v_n. Applying an iteration method to (2.14), we can show that v_n satisfies

$$\tilde{P}_n \le v_n \le 2\tilde{P}_n. \tag{2.26}$$

Recall v_n is defined by (2.7) and $\xi_n = x_n/z_n$. From assumption (A1), we can show that P^2/r^2, $\tilde{P}/r$, and $\tilde{P}^2/r^2 \in \mathcal{C}$, and hence, we may construct the following solution of (2.7):

$$\xi_n = \prod_{k=n}^{\infty} \frac{r_k}{r_k + P_k + v_k}, \qquad n \ge N, \tag{2.27}$$

which is clearly a principal solution of (2.3). A nonprincipal solution is defined by

$$\eta_n = \xi_n \sum_{k=N}^{n-1} \frac{1}{r_k \xi_k \xi_{k+1}}, \qquad n \ge N. \tag{2.28}$$

From (2.26), (2.27), and the given conditions, we can get the desired estimate for the principal solution

$$\xi_n = \exp(-Q_n + O(\varphi_n)), \tag{2.29}$$

and then estimate the nonprincipal solution η by (2.28) and (2.29).

Proof of Theorem 2.1. (ii). Since $\xi_n = x_n/z_n$ is a nonoscillatory solution of (2.3), $\xi_n \xi_{n+1} > 0$ for $n \ge N \ge 0$, (2.3) is nonoscillatory, and we may define $u_n = r_n \Delta\xi_n/\xi_n$, $u_n > -r_n$, for $n \ge N$. It is known [16, Lemma 2.2] that u_n satisfies

$$u_n = \sum_{k=n}^{\infty} \frac{u_k^2}{u_k + r_k} + P_n. \tag{2.30}$$

Obviously, $u_n/r_n = \Delta\xi_n/\xi_n \to 0$ because $\xi_n \to 1$. It follows from (2.30) that

$$\sum_{k=n}^{\infty} \frac{u_k^2}{r_k} = \sum_{k=n}^{\infty} \frac{u_k^2}{u_k + r_k}\left(1 - \frac{u_k}{r_k}\right) < \infty. \tag{2.31}$$

If we put $w_n = \log \xi_n$, then $w_n \to 0$ as $n \to \infty$ and $\Delta\xi_n/\xi_n = e^{\Delta w_n} - 1$. So,

$$\frac{u_n}{r_n} = \Delta w_n + \frac{(\Delta w_n)^2}{2} + O\left((\Delta w_n)^3\right). \tag{2.32}$$

From (2.31) and (2.32), we have $r(\Delta w)^2 \in \mathcal{C}$, which implies $(\Delta w)^2 \in \mathcal{C}$ because r_n is bounded away from zero. We then see that $u/r \in \mathcal{C}$ from $(\Delta w)^2 \in \mathcal{C}$ and (2.32).

Since $u_n = v_n + P_n$ and $P/r \in \mathcal{C}$, we get $v/r \in \mathcal{C}$, and hence, $v^2/r^2 \in \mathcal{C}$. Furthermore, from (2.32) and $(\Delta w_n)^2 \in \mathcal{C}$, it follows that $u^2/r^2 \in \mathcal{C}$, which in turn implies $P^2/r^2 \in \mathcal{C}$.

Since $P/r \in \mathcal{C}$ and $P^2/r^2 \in \mathcal{C}$, we can find two constants $0 < \alpha < \beta$ such that

$$\alpha \le q_n \le \beta, \qquad n \ge N, \tag{2.33}$$

$$\alpha \le r_k G(k,n) \le \beta, \qquad N \le n \le k. \tag{2.34}$$

On the other hand, v satisfies the second-level Riccati equation (2.14) and $0 \le \widetilde{P}_n \le v_n$. Thus, $\widetilde{P}/r \in \mathcal{C}$ since $v/r \in \mathcal{C}$. Denote $H_n := \sum_{k=n}^{\infty} P_k^2/r_k$. Now, with the aid of (2.34), we arrive at

$$\sum_{n=N}^{\infty} \frac{H_n}{r_n} = \sum_{n=N}^{\infty} \frac{1}{r_n} \sum_{k=n}^{\infty} \frac{P_k^2}{r_k} \le \sum_{n=N}^{\infty} \frac{1}{r_n} \sum_{k=n}^{\infty} \frac{1}{\alpha} G(k,n) P_k^2 = \frac{1}{\alpha} \sum_{n=N}^{\infty} \frac{\widetilde{P}_n}{r_n} < \infty.$$

Finally, by the definition of $\widetilde{P}_n$ and the boundedness of $r_k G(k,n)$, we obtain $P^2/r \in \mathcal{C}$. Then we have

$$\sum_{k=N}^{n} \frac{R_{k+1} P_k^2}{r_k} = \sum_{k=N}^{n} \frac{H_k}{r_k} + R_N H_N - R_{n+1} \sum_{k=n+1}^{\infty} \frac{P_k^2}{r_k},$$

which implies (A1) and ends the proof of Theorem 2.1(ii). ∎

3. AN EXAMPLE

In this section, we give an example illustrating the scope of improvement of our results.

Let h be a positive sequence such that $h_n \downarrow 0$ as $n \to \infty$. Then $\sum_{k=n}^{\infty} (-1)^n h_n$ converges and we want to estimate it. Let $b_n = (-1)^{n-1}/2$. Then $\Delta b_n = (-1)^n = 2b_{n+1}$. Since $b_n h_n$, $b_n \Delta h_n$, and $b_n \Delta^2 h_n$ all tend to zero, by (2.2), we have

$$\begin{aligned}
\sum_{k=n}^{\infty} (-1)^k h_k &= -b_n h_n - \sum_{k=n}^{\infty} b_{k+1} \Delta h_k = \frac{(-1)^n h_n}{2} - \frac{1}{2} \sum_{k=n}^{\infty} \Delta b_k \Delta h_k \\
&= (-1)^n \left(\frac{h_n}{2} - \frac{\Delta h_n}{4} \right) + \frac{1}{2} \sum_{k=n}^{\infty} b_{k+1} \Delta^2 h_k \\
&= (-1)^n \left(\frac{h_n}{2} - \frac{\Delta h_n}{4} + \frac{\Delta^2 h_n}{8} \right) - \frac{1}{4} \sum_{k=n}^{\infty} b_{k+1} \Delta^3 h_k.
\end{aligned} \tag{3.1}$$

In particular, if $h_n = n^{-\lambda}$, $\lambda > 0$, then

$$\begin{aligned}
\Delta h_n &= -\frac{\lambda}{n^{\lambda+1}} \left(1 - \frac{\lambda+1}{2n} + O\left(n^{-2}\right) \right), \\
\Delta^2 h_n &= \frac{\lambda(\lambda+1)}{n^{\lambda+2}} \left(1 - \frac{\lambda+2}{n} + O\left(n^{-2}\right) \right), \\
\Delta^3 h_n &= -\frac{\lambda(\lambda+1)(\lambda+2)}{n^{\lambda+3}} \left(1 + O\left(n^{-1}\right) \right),
\end{aligned}$$

so that, from (3.1),

$$\sum_{k=n}^{\infty} \frac{(-1)^k}{k^{\lambda}} = \frac{(-1)^n}{2n^{\lambda}} + \frac{(-1)^n \lambda}{4n^{\lambda+1}} + O\left(n^{-\lambda-3}\right). \tag{3.2}$$

EXAMPLE. Consider the difference equations

$$\Delta\left((-1)^n n^\alpha \Delta x_n\right) + \left(a_n + (-1)^n n^{-\beta}\right) x_{n+1} = 0, \qquad n \geq 1, \tag{3.3}$$

$$\Delta\left((-1)^n n^\alpha \Delta z_n\right) + a_n z_{n+1} = 0, \qquad n \geq 1, \tag{3.4}$$

where $a_n = (-1)^{n-1}((n+1)^\alpha - n^\alpha) - (n+1)^\alpha - n^\alpha$, $\alpha \leq 1$, $\beta > 0$. Since $z_n z_{n+1} = (-1)^n$, we easily see that (3.4) has a solution $z_n = (-1)^{[n/2]}$, where $[u]$ is the greatest integer less than or equal to u. Since $r_n = c_n z_n z_{n+1} = n^\alpha$ and $\sum^\infty 1/r_n = \infty$, z is a principal solution of (3.4). Now the transform $x_n = z_n \xi_n$ leads us to

$$\Delta\left(n^\alpha \Delta \xi_n\right) + (-1)^n n^{-\beta} \xi_{n+1} = 0, \qquad n \geq 1. \tag{3.5}$$

Ignoring the value of r_0 and the contribution of r_0 to R_n, we have that

$$R_n = \sum_{k=1}^{n-1} \frac{1}{r_k} = \frac{n^{1-\alpha}}{1-\alpha}\left(1 + O\left(n^{\alpha-1}\right)\right), \qquad \text{if } \alpha < 1, \tag{3.6}$$

$$R_n = \log n \left(1 + O\left(\frac{1}{\log n}\right)\right), \qquad \text{if } \alpha = 1. \tag{3.7}$$

We first assume $\alpha < 1$. It follows from (3.2) that

$$P_n = \sum_{k=n}^{\infty} \frac{(-1)^k}{k^\beta} = \frac{(-1)^n}{2n^\beta}\left(1 + \frac{\beta}{2n} + O\left(n^{-3}\right)\right). \tag{3.8}$$

If $\alpha + \beta > 0$, then $P/r \in C$ and, by (3.2) again,

$$Q_n = \sum_{k=n}^{\infty} \frac{P_k}{r_k} = \frac{(-1)^n}{4n^{\alpha+\beta}}\left(1 + \frac{\alpha+2\beta}{2n} + O\left(n^{-2}\right)\right). \tag{3.9}$$

From (3.6) and (3.8), we see that

$$\frac{R_{k+1}P_k^2}{r_k} = (C + o(1))n^{1-2\alpha-2\beta},$$

so that (A1) holds if $\alpha + \beta > 1$, and if so,

$$\varphi_n = \sum_{k=n}^{\infty} \frac{R_{k+1}P_k^2}{r_k} = (C + o(1))n^{2-2\alpha-2\beta}, \tag{3.10}$$

$$Q_n^2 = O\left(n^{-2\alpha-2\beta}\right) = o(\varphi_n), \tag{3.11}$$

where C is a positive constant, not necessarily the same at each appearance. Now by Theorem 2.1, from (3.9) and (3.11), we obtain that if $\alpha < 1$, $\beta > 0$, and $\alpha + \beta > 1$, then (3.3) has a principal solution x satisfying

$$x_n = (-1)^{[n/2]}\left(1 - \frac{(-1)^n}{4n^{\alpha+\beta}} - \frac{(-1)^n(\alpha+2\beta)}{8n^{\alpha+\beta+1}} + O\left(n^{-\gamma}\right)\right), \tag{3.12}$$

where $\gamma = \min\{\alpha+\beta+2, 2\alpha+2\beta-2\}$, and from (2.17), (3.6) and (3.8)–(3.11),

$$\Delta\left((-1)^{[n/2]}x_n\right) = \frac{(-1)^n}{2n^{\alpha+\beta}} + \frac{(-1)^n\beta}{4n^{\alpha+\beta+1}} + O\left(\frac{1}{n^{2\alpha+2\beta-1}}\right).$$

On the other hand, we observe that $\overline{Q}_n = O(n^{-\alpha-\beta-1}) = \overline{Q}_{n+1}$ from (3.9) and that $\overline{Q_n^2} = o(\overline{\varphi}_n)$ from (3.11). Moreover, from (3.10), $\varphi_n/r_n = (C+o(1))n^{2-3\alpha-2\beta}$. Therefore, there are three cases to deal with for the estimation of $\overline{\varphi}_n$, that is,

$$\overline{\varphi}_n = n^{\alpha-1}\log n\,(C+o(1)), \qquad \text{if } 3\alpha+2\beta-2=1, \tag{3.13a}$$

$$\overline{\varphi}_n = n^{\alpha-1}\,(C+o(1)), \qquad \text{if } 3\alpha+2\beta-2>1, \tag{3.13b}$$

$$\overline{\varphi}_n = n^{2-2\alpha-2\beta}\,(C+o(1)), \qquad \text{if } 3\alpha+2\beta-2<1. \tag{3.13c}$$

We remark here that $2\alpha+2\beta-2<1-\alpha$ when $3\alpha+2\beta-2<1$ and $R_n = O(\overline{\varphi}_n)$ in any case. Now from (2.18) and (3.9), we see (3.3) has a nonprincipal solution

$$y_n = (-1)^{[n/2]}R_n\left(1-(-1)^n4^{-1}n^{-\alpha-\beta}+O\left(n^{-\alpha-\beta-1}\right)+O\left(\overline{\varphi}_n\right)\right),$$

where $\overline{\varphi}_n$ has an order given by one of the relations (3.13) according to the values of α and β.

Because $R_nP_n = O(n^{1-\alpha-\beta})$, (A2) is valid when $\alpha+\beta>1$. Hence, by Theorem 2.2, taking $Q_n = o(n^{1-\alpha-\beta})$ into account, we obtain

$$\Delta\left((-1)^{[n/2]}y_n\right) = n^{-\alpha}\left(1+O\left(n^{1-\alpha-\beta}\right)\right).$$

Finally, from (3.6) and (3.8) it is clear that (A3) and (A4) hold if $2\alpha+\beta>1$ and that (A5) holds if $3\alpha+2\beta>3$. If so, using (3.2), we get

$$\psi_n = \sum_{k=n}^{\infty}\frac{R_{k+1}^2P_k^2}{r_k} = \sum_{k=n}^{\infty}\frac{C+o(1)}{n^{3\alpha+2\beta-2}} = O\left(n^{3-3\alpha-2\beta}\right). \tag{3.14}$$

From (3.6) and (3.9), we have

$$R_nQ_n = \frac{(-1)^n}{4(1-\alpha)n^{2\alpha+\beta-1}}+\frac{(-1)^n(\alpha+2\beta)}{8(1-\alpha)n^{2\alpha+\beta}}+O\left(n^{-2\alpha-\beta-1}\right). \tag{3.15}$$

Since $(Q_n+Q_{n+1})/r_n = 2Q_n/r_n - P_n/r_n^2 = (-1)^n(\alpha+\beta)n^{-2\alpha-\beta-1}(1/4+O(1/n))$, we have $S_n + T_n = \sum_{k=n}^{\infty}(Q_k+Q_{k+1})/r_k = O(n^{-2\alpha-\beta-1})$. Note that the term $o(\|RQ\|_n)$ in (2.24) is a part of $\sum_{k=n}^{\infty}(Q_k+Q_{k+1})^2/r_k$, and hence, it can be treated as $O(\psi_n)$ by (3.11). Thus, by Theorem 2.3, from (3.14), (3.15), and (2.24), we conclude that if $\alpha<1$, $\beta>0$, $2\alpha+\beta>1$, and $3\alpha+2\beta>3$, then the nonprincipal solution y satisfies

$$y_n = (-1)^{[n/2]}\left(R_n+L-\frac{(-1)^n}{4(1-\alpha)n^{2\alpha+\beta-1}}-\frac{(-1)^n(\alpha+2\beta)}{8(1-\alpha)n^{2\alpha+\beta}}+O\left(\frac{1}{n^\sigma}\right)\right),$$

where L is a constant and $\sigma=\min\{2\alpha+\beta+1, 3\alpha+2\beta-3\}$.

For comparison of condition (2.25) used in [13] with ours, let $\alpha=0$ and consider equation (3.5) only. Then for (2.25) to hold, we need $\beta>3$, while for assumptions (A1)–(A5) we need only $\beta>3/2$.

The case $\alpha=1$ can be handled in a similar way and we omit the details.

4. A POINCARÉ DIFFERENCE EQUATION

The m^{th}-order difference equation [17]

$$x_{n+m}+\left(t_1+p_n^{(1)}\right)x_{n+m-1}+\cdots+\left(t_m+p_n^{(m)}\right)x_n=0, \qquad n\ge 0, \tag{4.1}$$

where t_i, $1\le i\le m$, are constants with $t_m\ne 0$, has been studied by many authors (see [18,19]). Most authors suppose that the polynomial

$$\gamma(\lambda) = \lambda^m + t_1\lambda^{m-1}+\cdots+t_m$$

has distinct zeros $\lambda_1, \lambda_2, \ldots, \lambda_m$. Various smallness conditions on the coefficients $p_n^{(i)}$, $i = 1, \ldots, m$, are found so that the fundamental solutions of (4.1) behave in some sense like λ_i^n, $i = 1, \ldots, m$, as $n \to \infty$. The case where $\gamma(\lambda)$ has multiple zeros has rarely been investigated.

Chen and Wu [14] consider the second-order Poincaré difference equation

$$x_{n+2} - t(2 + b_n)x_{n-1} + t^2(1 + c_n)x_n = 0, \qquad t \neq 0, \tag{4.2}$$

where b_n, c_n, and x_n are all real numbers. The unperturbed equation of (4.2) is

$$z_{n+2} - 2tz_{n+1} + t^2 z_n = 0, \tag{4.3}$$

whose characteristic polynomial has a double root $\lambda = t$. Note that (4.3) has two linearly independent solutions $z_n = t^n$ and nt^n, and $z_n = t^n$ is principal when $t > 0$.

Let $x_n = t^n \xi_n$. Then (4.2) is reduced to

$$\xi_{n+2} - (2 + b_n)\xi_{n+1} + (1 + c_n)\xi_n = 0. \tag{4.4}$$

We assume that $c_n \neq -1$ for $n \geq 0$, and set

$$r_0 = 1, \quad r_n = \prod_{k=0}^{n-1} \frac{1}{1 + c_k}, \qquad \text{for } n \geq 1. \tag{4.5}$$

Multiplying both sides of (4.3) by r_{n+1} leads us to a self-adjoint difference equation

$$\Delta(r_n \Delta \xi_n) + p_n \xi_{n+1} = 0, \qquad n \geq 0, \tag{4.6}$$

where

$$p_n = r_{n+1}(c_n - b_n), \qquad n \geq 0. \tag{4.7}$$

Now, all that is needed is to apply results in Section 2 to (4.6).

In [14], Chen and Wu use a stronger assumption that b, c, and $c^2 \in \mathcal{C}$, and $|c_n| < 1$. This is mainly because the coefficient r is required to be bounded in [14].

We first assume that $c_n > -1$, $n \geq 0$. Then $r_n > 0$. Recall that an infinite product $\prod_{k=0}^{\infty}(1+c_k)$ of positive factors is called convergent if $\prod_{k=0}^{n-1}(1 + c_k)$ tends to a positive number as $n \to \infty$. It is known that if c, $c^2 \in \mathcal{C}$, then $\prod_{k=0}^{\infty}(1 + c_k)$ is convergent. Moreover, from $\log r_n = -\sum_{k=0}^{n-1} \log(1 + c_k)$, we see that if

$$\liminf_{n \to \infty} \sum_{k=0}^{n-1} c_k > -\infty \quad \text{and} \quad \sum_{n=0}^{\infty} c_n^2 < \infty, \tag{4.8}$$

then r_n is bounded above, and that if

$$\limsup_{n \to \infty} \sum_{k=0}^{n-1} c_k < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} c_n^2 < \infty, \tag{4.9}$$

then r_n is bounded below, i.e., $r_n \geq m > 0$ with m a constant. We also see that (4.8) $\Rightarrow$ $\sum^{\infty} 1/r_n = \infty$ and (4.9) $\Rightarrow R_n = O(n)$ as $n \to \infty$.

Now we have the following results.

THEOREM 4.1. *Suppose that $c_n > -1$ for $n \geq 0$, $\sum^{\infty} 1/r_n = \infty$, $p \in \mathcal{C}$, and $P/r \subset \mathcal{C}$. If (A1) holds, then (4.2) has two linearly independent solutions x and y satisfying (2.16)–(2.18) with $z_n = t^n$. Conversely, if (4.9) holds, or more generally, if r_n is bounded below, and if (4.2) has a solution x such that $x_n t^{-n} \to 1$ as $n \to \infty$, then (A1) holds.*

THEOREM 4.2. *In Theorem 4.1, if in addition, (A2) holds, then the solution y of (4.2) satisfies (2.22) with $z_n = t^n$. Conversely, if (4.2) has a solution y such that $r_n\Delta(y_n t^{-n}) \to 1$ as $n \to \infty$, then (A2) holds.*

THEOREM 4.3. *Suppose that $c_n > -1$ for $n \geq 0$, $\sum^\infty 1/r_n = \infty$, $p \in \mathcal{C}$, and $P/r \in \mathcal{C}$. If (A3)–(A5) hold, then (4.2) has two linearly independent solutions x and y, respectively, satisfying (2.16) and (2.24) with $z_n = t^n$.*

Theorems 4.1–4.3 follow from Theorems 2.1–2.3, respectively, and their proofs are omitted.

In the case in which $\sum^\infty 1/r_n < \infty$, we have to make another transformation. Let $s_n = \sum_{k=n}^\infty 1/r_k$. Since s_n is a principal solution of the nonoscillatory equation $\Delta(r_n\Delta\xi_n) = 0$, $\sum^\infty 1/(r_n s_n s_{n+1}) = \infty$. Let $\xi_n = s_n\zeta_n$. Then we have

$$\Delta(\rho_n\Delta\zeta_n) + q_n\zeta_{n+1} = 0, \tag{4.10}$$

where $q_n = s_{n+1}^2 p_n$, $\rho_n = r_n s_n s_{n+1}$ with $\sum^\infty 1/\rho_n = \infty$. Now we can study (4.10) instead of (4.6) to obtain results similar to the above theorems. We omit the details.

Finally, we assume that $c_n \neq -1$, and r_n oscillates. In order to apply the results in Section 2, we must find an eventually disconjugate equation, say,

$$\Delta(r_n\Delta\zeta_n) + q_n\zeta_{n+1} = 0, \tag{4.11}$$

so that $p_n - q_n$ is in some sense small. Doing so is actually equivalent to properly transforming (4.6) into a self-adjoint equation with a positive coefficient in the leading term. But we do not pursue this here.

5. NONLINEAR PERTURBATIONS

Suppose that the eventually disconjugate equation (1.2) is perturbed both linearly and nonlinearly

$$\Delta(c_n\Delta x_n) + (a_n + f_n)x_{n+1} = g(n, x_n, x_{n+1}). \tag{5.1}$$

Let z be the principal solution of (1.2). The transformation $x_n = z_n u_n$ leads us to

$$L_n u := \Delta(r_n\Delta u_n) + p_n u_{n+1} = q(n, u_n, u_{n+1}), \tag{5.2}$$

where $r_n = c_n z_n z_{n+1}$, $p_n = f_n z_{n+1}^2$, and

$$q(n, u_n, u_{n+1}) = z_{n+1} g(n, z_n u_n, z_{n+1} u_{n+1}). \tag{5.3}$$

We view (5.2) as a nonlinear perturbation of

$$L_n\xi = \Delta(r_n\Delta\xi_n) + p_n\xi_{n+1} = 0. \tag{2.3}$$

Approximation for solutions of (5.1) via solutions of (1.2) is equivalent to the question: asymptotic to a given solution of (2.3), find a solution of (5.2). The idea we will introduce stems from the variation of constant method. Let ξ and η be a fundamental system of solutions of (2.3), namely, they are linearly independent and

$$r_n(\xi_n\eta_{n+1} - \xi_{n+1}\eta_n) = 1, \qquad n \geq 0. \tag{5.4}$$

From (5.2) and (2.3), we have

$$\xi_{n+1}L_n u - u_{n+1}L_n\xi = \Delta(r_n\xi_n u_{n+1} - r_n\xi_{n+1}u_n) = \xi_{n+1}q(n, u_n, u_{n+1}). \tag{5.5}$$

If the right-hand side of (5.5) is summable, we then get

$$r_n\xi_n u_{n+1} - r_n\xi_{n+1}u_n = B - \sum_{k=n}^{\infty} \xi_{k+1}q(k, u_k, u_{k+1}), \tag{5.6}$$

where B is a constant. Similarly,

$$r_n\eta_n u_{n+1} - r_n\eta_{n+1}u_n = -A - \sum_{k=n}^{\infty} \eta_{k+1}q(k, u_k, u_{k+1}) \tag{5.7}$$

with A a constant. Multiplying (5.6) and (5.7) by η_n and ξ_n, respectively, and subtracting the resulting equations, with the aid of (5.4), we then arrive at a formal "integral" equation

$$u_n = A\xi_n + B\eta_n + \sum_{k=n}^{\infty} (\xi_n\eta_{k+1} - \eta_n\xi_{k+1})q(k, u_k, u_{k+1}). \tag{5.8}$$

Now, given any solution $w_n = A\xi_n + B\eta_n$ of (2.3), consider the equation

$$\zeta_n = \sum_{k=n}^{\infty} (\xi_n\eta_{k+1} - \eta_n\xi_{k+1})q(k, w_k + \zeta_k, w_{k+1} + \zeta_{k+1}). \tag{5.9}$$

It is easy to verify that if (5.9) has a solution ζ_n that is expressed by a convergent series, then $u_n = w_n + \zeta_n$ is a solution of (5.2).

For existence of a solution of (5.9), we need the following assumption.

(A6) For any u, v, u_1, v_1, $w = \max\{|u|, |v|\}$, $w_1 = \max\{w, |u_1|, |v_1|\}$, there exist two nonnegative sequences β and γ such that for $n \geq 0$,

$$|q(n, u, v)| \leq \beta_n + \gamma_n h(w), \tag{5.10}$$

$$|q(n, u, v) - q(n, u_1, v_1)| \leq \gamma_n h(w_1)(|u - u_1| + |v - v_1|), \tag{5.11}$$

and

$$B_n := \sum_{k=n}^{\infty} R_k\beta_k < \infty, \qquad \Gamma_n := \sum_{k=n}^{\infty} R_k h(R_{k+1})\gamma_k < \infty, \tag{5.12}$$

where h is a nonnegative and nondecreasing continuous function on $[0, \infty)$ satisfying $h(au) \leq C_a h(u)$ with C_a a constant depending only on a.

THEOREM 5.1. *If $p \in \mathcal{C}$, $P/r \in \mathcal{C}$, (A1) and (A6) hold, then for any constants A and B, there exists a unique solution $x = \{x_n\}_{n=N}^{\infty}$ of (5.1) for sufficiently large N such that, for $n \geq N$,*

$$\begin{aligned} x_n = {} & Az_n\left(1 - Q_n + Q_n^2\left(\frac{1}{2} + o(1)\right) + O(\varphi_n)\right) \\ & + Bz_nR_n\left(1 - Q_n + \overline{Q}_n + \overline{Q}_{n+1} + O\left(\overline{Q_n^2} + \overline{\varphi}_n\right)\right) + z_nO(B_n + \Gamma_n). \end{aligned} \tag{5.13}$$

PROOF. By Theorem 2.1, (2.3) is nonoscillatory and has a principal solution ξ and a nonprincipal solution η satisfying (5.4) and

$$\xi_n = 1 - Q_n + Q_n^2\left(\frac{1}{2} + o(1)\right) + O(\varphi_n),$$

$$\eta_n = R_n\left(1 - Q_n + \overline{Q}_n + \overline{Q}_{n+1} + O\left(\overline{Q_n^2} + \overline{\varphi}_n\right)\right).$$

Let $w_n = A\xi_n + B\eta_n$ and consider equation (5.9). We claim that (5.9) has a unique solution $\zeta = \{\zeta_n\}_{n=N}^{\infty}$ for sufficiently large N.

Let X be the Banach space of all bounded sequences $z = \{z_n\}_{n=N}^{\infty}$ equipped with the supremum norm $\|z\| = \sup_{n\geq N} |z_n|$, where N is an undetermined integer. Define an operator T on X by

$$(T\zeta)_n = \sum_{k=n}^{\infty} (\xi_n\eta_{k+1} - \eta_n\xi_{k+1})q(k, w_k + \zeta_k, w_{k+1} + \zeta_{k+1}), \qquad n \geq N. \tag{5.14}$$

Since $\xi_n = O(1)$, $\eta_n = O(R_n)$, we know that $w_n = O(R_n)$ and $w_n + \zeta_n = O(R_n)$ for any $\zeta \in X$. Consequently, there is a constant K_1 such that $|\xi_n\eta_{k+1} - \eta_n\xi_{k+1}| \leq K_1 R_k$ for $k \geq n$. Thus, from (5.10) and (5.12),

$$\begin{aligned} |(T\zeta)_n| &\leq K_1 \sum_{k=n}^{\infty} R_k(\beta_k + \gamma_k h(O(R_{k+1}))) \\ &\leq K_2 \sum_{k=n}^{\infty} (R_k\beta_k + R_k h(R_{k+1})\gamma_k) = K_2(B_n + \Gamma_n), \end{aligned} \tag{5.15}$$

where K_2 is a constant. Therefore, $T\zeta \in X$.

Next, for any $u, v \in X$, from (5.11), we have

$$\begin{aligned} |(Tu)_n - (Tv)_n| &\leq K_1 \sum_{k=n}^{\infty} R_k|q(k, w_k + u_k, w_{k+1} + u_{k+1}) \\ &\qquad - q(k, w_k + v_k, w_{k+1} + v_{k+1})| \\ &\leq K_3 \sum_{k=n}^{\infty} R_k\gamma_k(|u_k - v_k| + |u_{k+1} - v_{k+1}|)h(R_{k+1}) \\ &\leq 2K_3\Gamma_N\|u - v\|, \qquad n \geq N, \end{aligned} \tag{5.16}$$

which implies that $T : X \to X$ is continuous.

Now we choose N large enough so that $2K_3\Gamma_N < 1$, and hence, from (5.16), $T : X \to X$ is a contraction. Therefore, T has a unique fixed point $\zeta \in X$ which is a solution of (5.9) for $n \geq N$, and, by (5.15),

$$\zeta_n = O(B_n + \Gamma_n). \tag{5.17}$$

Since $u = w + \zeta$ is a solution of (5.2), we obtain a solution $x_n = z_n u_n$ of (5.1) satisfying (5.13) for $n \geq N$ by (5.17). The proof of Theorem 5.1 is complete. ∎

THEOREM 5.2. *Suppose that $p \in \mathcal{C}$, $P/r \in \mathcal{C}$. If (A3)–(A6) hold, then for any constant A and B, there exists a unique solution $x = \{x_n\}_{n=N}^{\infty}$ of (5.1) for sufficiently large N such that*

$$x_n = z_n(A + BR_n + o(1)). \tag{5.18}$$

Theorem 5.2 follows from Theorem 2.3, by a similar argument except that we choose $w = (A - BL)\xi + B\eta$ instead, where L is the constant in (2.24). The term $o(1)$ in (5.18) can also be specified more precisely. We omit the details.

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